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On the Theory of the Tides.

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On the Theory of the Tides,

by

KINNOSUKE OGURA, Ôsaka.

1. In Hough's theory of the tides [1] we see that the admissible values of f for the case of free oscillation are the *real* roots of the function defined by the continued fraction

$$(\mathrm{I}) \qquad L_1 - \frac{\dfrac{1}{3.5^2.7}}{L_3 -} \; \frac{\dfrac{1}{7.9^2.11}}{L_5 -} \cdots\cdots\cdots$$

or

$$(\mathrm{II}) \qquad L_2 - \frac{\dfrac{1}{5.7^2.9}}{L_4 -} \; \frac{\dfrac{1}{9.11^2.13}}{L_6 -} \cdots\cdots\cdots,$$

where

$$L_n \equiv \frac{f^2-1}{n(n+1)} + \frac{2}{(2n-1)(2n+3)} - \frac{1}{\beta}\left(1 - \frac{3}{2n+1}\,\frac{\rho}{\rho_0}\right),$$

β, ρ, ρ_0 being real positive constants and $\rho < \rho_0$.

Hough calculated the approximate values of the roots; but it will be desirable, from the purely mathematical standpoint, to discuss *the convergency of these continued fractions,* and to prove *the existence of the real roots.*

Now consider the continued fraction

$$(1) \qquad a_1 z + b_1 - \frac{c_1}{a_2 z + b_2 -} \; \frac{c_2}{a_3 z + b_3 -} \; \frac{c_3}{a_4 z + b_4 -} \cdots\cdots,$$

where a_ν, b_ν, c_ν $(\nu = 1, 2, \cdots\cdots)$ are real constants such that

$$a_\nu > 0 \ (\nu = 1, 2, \cdots\cdots), \quad \lim a_\nu = 0;$$

$$\lim b_\nu = \text{negative and finite } (=b, \text{ say});$$

$$c_\nu > 0 \ (\nu = 1, 2, \cdots\cdots), \quad \lim c_\nu = 0,$$

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(1) S. Hough. "On the application of harmonic analysis to the dynamical theory of the tides," Phil. Trans. London, 189 (1897), p. 201. See also H. Lamb, Hydrodynamics, 4. ed. (1916), p. 335, where the physical meanings of f, β, ρ and ρ_0 will be found.

$$\lim \frac{c_2 c_4 \cdots\cdots c_{2\nu-2}}{c_1 c_3 \cdots\cdots c_{2\nu-3}} \cdot \frac{1}{c_{2\nu-1}} = +\infty,$$

$$\lim \frac{c_1 c_3 \cdots\cdots c_{2\nu-3}}{c_2 c_4 \cdots\cdots c_{2\nu-2}} \cdot \frac{c_{2\nu-1}}{c_{2\nu}} = +\infty.$$

The continued fraction (1) contains (I) and (II) as particular cases. For, if we put

$$z = f^2$$

$$a_\nu = \frac{1}{2\nu(2\nu-1)},$$

$$b_\nu = \frac{2}{(4\nu-3)(4\nu+1)} - \frac{1}{2\nu(2\nu-1)} - \frac{1}{\beta}\left(1 - \frac{3}{4\nu-1}\,\frac{\rho}{\rho_1}\right),$$

$$c_\nu = \frac{1}{(4\nu-1)(4\nu+1)^2(4\nu+3)}, \quad (\nu = 1, 2, 3, \cdots\cdots).$$

We have

$$\frac{c_2 c_4 \cdots\cdots c_{2\nu-2}}{c_1 c_3 \cdots\cdots c_{2\nu-3}} \cdot \frac{1}{c_{2\nu-1}} = 3.5^2.(8\nu-1)\left[\frac{13.21.\cdots\cdots(8\nu-3)}{9.17.\cdots\cdots(8\nu-7)}\right]^2 \quad (\nu \geqq 2),$$

$$\frac{c_1 c_3 \cdots\cdots c_{2\nu-3}}{c_2 c_4 \cdots\cdots c_{2\nu-2}} \cdot \frac{c_{2\nu-1}}{c_{2\nu}} = \frac{8\nu+3}{3}\left[\frac{9.17.\cdots\cdots(8\nu+1)}{5.13.\cdots(8\nu-3)}\right]^2, \quad (\nu \geqq 1),$$

and then (1) becomes (I). Also by a similar substitution (1) becomes (II).

Consequently in this note I will deal with the continued fraction (1) only.

Convergence and analytic character of the continued fraction.

2. In order to discuss the convergency of the continued fraction (1) we use the following theorem[1]: Let

$$q_0 + \frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} + \cdots\cdot$$

be a continued fraction whose elements p_ν and q_ν are functions of the complex variable z within a certain domain T, such that the sequences (p_ν) and (q_ν) converge uniformly to p and q respectively within T; and let ρ_1, ρ_2 be the roots of the quadratic equation

[1] O. Perron, Die Lehre von den Kettenbrüchen (1913), p. 285.

$$\rho^2 - q\rho - p = 0.$$

If there exist three positive constants k, k', k'' which satisfy the inequalities

$$k \leqq |\rho_1| \leqq k', \qquad \left|\frac{\rho_2}{\rho_1}\right| \leqq k'', \qquad k'' < 1,$$

then we can find the positive integer m such that

$$q_\nu + \frac{p_{\nu+1}}{q_{\nu+1}} + \frac{p_{\nu+2}}{q_{\nu+2}} + \cdots\cdots, \qquad \nu \geqq m$$

converges uniformly within the domain T.

If we put

$$p_\nu = -c_\nu, \quad q_\nu = a_{\nu+1}z + b_{\nu+1}$$

and take T as the circle $|z| < R$ where R is any positive quantity, (p_ν) and (q_ν) converge uniformly to 0 and b respectively within T; and

$$\rho_1 = b, \qquad \rho_2 = 0.$$

Therefore we can choose the positive integer m for which

$$(2) \qquad a_{m+1}z + b_{m+1} - \frac{c_{m+1}}{a_{m+2}z + b_{m+2}} - \frac{c_{m+2}}{a_{m+3}z + b_{m+3}} - \cdots\cdots$$

converges uniformly for $|z| < R$.

Denote by $A_{m,n}(z)/B_{m,n}(z)$ the nth convergent of (2). Then the series of rational functions

$$\frac{A_{m,1}(z)}{B_{m,1}(z)} + \left[\frac{A_{m,2}(z)}{B_{m,2}(z)} - \frac{A_{m,1}(z)}{B_{m,1}(z)}\right] + \left[\frac{A_{m,3}(z)}{B_{m,3}(z)} - \frac{A_{m,2}(z)}{B_{m,2}(z)}\right] + \cdots\cdots$$

converges uniformly and $B_{m,n}(z) \neq 0$ $(n = 1, 2, \cdots\cdots)$ for $|z| < R$. Consequently it follows by Weierstrass' theorem, that the series and therefore the continued fraction (2) is a regular analytic function of z for $|z| < R$. This result is true how large R may be, so that the continued fraction (2) must represent an *integral function*, $\Phi_m(z)$, say.

3. If $K_n(z) \equiv \dfrac{A_n(z)}{B_n(z)}$ be the nth convergent of (1), this continued fraction converges uniformly to

$$(3) \qquad K(z) \equiv \frac{A_{m-1}(z)\,\Phi_m(z) - c_m A_{m-2}(z)}{B_{m-1}(z)\,\Phi_m(z) - c_m B_{m-2}(z)}$$

within *the domain* T' which is the z-plane excluded by the circles round the roots $\zeta_1, \zeta_2, \zeta_3, \cdots$ of the denominator

$$(4) \qquad B_{m-1}(z)\,\Phi_m(z) - c_m B_{m-2}(z)$$

with radii sufficiently small.

Now the function (4) does not vanish identically. For, if it vanish identically, we must have

$$\Phi_m(z) \equiv c_m \frac{B_{m-2}(z)}{B_{m-1}(z)}.$$

But $\Phi_m(z)$ is an integral function, while $B_{m-2}(z)/B_{m-1}(z)$ is a proper fraction, the degree of the polynomial $B_{m-1}(z)$ being higher than that of $B_{m-2}(z)$; so that the above identity can not exist.

Next the denominator and the numerator of (3) have no common root. If α be such a root, then

$$A_{m-1}(\alpha)\, B_{m-2}(\alpha) - B_{m-1}(\alpha)\, A_{m-2}(\alpha) = 0,$$

which leads to a contradiction, because we have the well known identity

$$A_{m-1}(z)\, B_{m-2}(z) - B_{m-1}(z)\, A_{m-2}(z) \equiv (-1)^{2m-3} c_1 c_2 \cdots\cdots c_{m-1}.$$

Therefore we have obtained the theorem:

Theorem I. *The continued fraction* (1) *converges uniformly to the meromorphic function* $K(z)$ *within the domain* T'.

Existence of the positive roots.

4. The reciprocal of the continued fraction (1), i. e.

$$\frac{1}{a_1z+b_1} - \frac{c_1}{a_2z+b_2} - \frac{c_2}{a_3z+b_3} - \cdots\cdots$$

may be written

$$\frac{1}{a_1z+b_1} - \frac{1}{\frac{1}{c_1}(a_2z+b_2)} - \frac{1}{\frac{c_1}{c_2}(a_3z+b_3)} - \cdots\cdots$$

$$(5) \qquad - \frac{1}{\frac{c_2 c_4 \cdots\cdots c_{2\nu-2}}{c_1 c_3 \cdots\cdots c_{2\nu-3}} \cdot \frac{1}{c_{2\nu-1}}(a_{2\nu}z+b_{2\nu})}$$

$$- \frac{1}{\frac{c_1 c_3 \cdots\cdots c_{2\nu-3}}{c_2 c_4 \cdots\cdots c_{2\nu-2}} \cdot \frac{c_{2\nu-1}}{c_{2\nu}}(a_{2\nu+1}z+b_{2\nu+1})} - \cdots\cdots$$

If $\dfrac{1}{K_\nu(z)} \equiv \dfrac{Q_\nu(z)}{P_\nu(z)}$ [1] be the νth convergent of (5), we have

[1] Of course we have $\dfrac{P_\nu(z)}{Q_\nu(z)} \equiv \dfrac{A_\nu(z)}{B_\nu(z)}$, $(\nu=1, 2, \cdots)$

$$P_{\nu-1}(z)\, Q_\nu(z) - Q_{\nu-1}(z)\, P_\nu(z) \equiv \pm 1;$$

so that $P_\nu(z)$ *and* $Q_\nu(z)$ *have no common root.*

Now we have

$$P_1(z) = a_1z + b_1,$$

$$P_2(z) = \frac{1}{c_1}(a_2z+b_2)(a_1z+b_1) - 1,$$

$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$$

$$P_\nu(z) = \lambda_\nu(z)\, P_{\nu-1}(z) - P_{\nu-2}(z),$$

$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots,$$

where

$$\lambda_2(z) = \frac{1}{c_1}(a_2z+b_2),$$

$$\lambda_{2\nu}(z) = \frac{c_2 c_4 \cdots\cdots c_{2\nu-2}}{c_1 c_3 \cdots\cdots c_{2\nu-3}} \cdot \frac{1}{c_{2\nu-1}}(a_{2\nu}z+b_{2\nu}), \quad (\nu \geqq 2),$$

$$\lambda_{2\nu+1}(z) = \frac{c_1 c_3 \cdots\cdots c_{2\nu-3}}{c_2 c_4 \cdots\cdots c_{2\nu-2}} \cdot \frac{c_{2\nu-1}}{c_{2\nu}}(a_{2\nu+1}z+b_{2\nu+1}), \quad (\nu \geqq 1);$$

whence we may take

$$(6) \qquad \begin{cases} P_0(z) = 1, \\ P_1(z) = a_1z + b_1, \\ P_\nu(z) = \lambda_\nu(z) P_{\nu-1}(z) - P_{\nu-2}(z), \quad (\nu = 2, 3, \cdots\cdots). \end{cases}$$

5. I. *For any particular value of* z, *real or complex, the sequence* $|P_\nu(z)|$ *is ultimately either monotonously increasing or monotonously decreasing.*

Since

$$\lim \lambda_\nu(z) = -\infty$$

for any particular value of z, we may take the positive integer μ such that

$$(7) \qquad |\lambda_\nu(z)| > 2, \qquad \nu \geqq \mu.$$

If $|P_\mu(z)|$, $|P_{\mu+1}(z)| \cdots\cdots$ decrease monotonously, our theorem is granted. If not, there exists $P_k(z)$, $k \geqq \mu$, such that $|P_k(z)| \geqq |P_{k-1}(z)|$; then $P_k(z) \neq 0$. From (6) and (7) we have

$$|P_{k+1}(z)| > 2\,|P_k(z)| - |P_{k-1}(z)|$$
$$= |P_k(z)| + (|P_k(z)| - |P_{k-1}(z)|) \geqq |P_k(z)|;$$

and similarly

$$| P_{k+n}(z) | > | P_{k+n-1}(z) |, \qquad (n=2, 3, \cdots\cdots).$$

II. *For any particular value of z, real or complex, the sequence $P_\nu(z)$ either approaches zero or becomes infinite.*

If $| P_\nu(z) | < G$ for all ν, then from

$$\lambda_{\nu+1}(z)\, P_\nu(z) = P_{\nu-1}(z) + P_{\nu+1}(z),$$

we have

$$| \lambda_{\nu+1}(z) | \,.\, | P_\nu(z) | < 2\,G.$$

Since $\lim | \lambda_\nu(z) | = +\infty$ we obtain

$$\lim | P_\nu(z) | = 0.$$

If $| P_\nu(z) |$ do not remain finite, then by I., $| P_\nu(z) |$ increases indefinitely.

6. Now the sequence of polynomials $P_\nu(z)$ $(\nu=0, 1, 2, \cdots\cdots)$ has the Sturm properties:

1) $P_0(z)=1$ $(\neq 0)$;
2) $P_\nu(z)$ is of the νth degree in z;
3) $P_\nu(z)$ and $P_{\nu+1}(z)$ are relatively prime $(\nu=1, 2, \cdots\cdots)$;
4) $P_{\nu+1}(a)\, P_{\nu-1}(a) < 0$, provided a is a real root of $P_\nu(z)=0$.

Hence the number of real roots of $P_\nu(z)=0$ in the interval $a \leqq z \leqq b$, where a and b are real and $P_\nu(a) \neq 0$, $P_\nu(b) \neq 0$, is at least $| V_\nu(a) - V_\nu(b) |$, $V_\nu(z_0)$ being the number of variations in sign of the sequence

$$P_0(z_0),\ P_1(z_0),\ P_2(z_0), \cdots\cdots,\ P_\nu(z_0).$$

Since

$$V_\nu(-\infty)=\nu,\quad V_\nu(+\infty)=0,$$

$P_\nu(z)=0$ *has at least ν distinct, real roots* and therefore exactly ν. Therefore $V_\nu(z)$ must decrease by unity each time when increasing real variable z passes through a (real) root of $P_\nu(z)=0$; and we can obtain the following theorems immediately:

The number of the real roots of $P_\nu(z)=0$ in the interval $a \leqq z \leqq b$, where $P_\nu(a) \neq 0$, $P_\nu(b) \neq 0$, is given by $V_\nu(a) - V_\nu(b)$;

The ν roots of $P_\nu(z)=0$ are separated by the $\nu-1$ roots of $P_{\nu-1}(z)=0$.

7. Let us take the positive integer N such that

$$\lambda_\nu(z_0) < -2, \quad \nu \geqq N.$$

If $P_{\nu-2}(z_0) > 0$ and $P_{\nu-1}(z_0) > 0$, then it follows from

$$P_\nu(z_0) = \lambda_\nu(z_0)\, P_{\nu-2}(z_0) - P_{\nu-2}(z_0)$$

that

$$P_\nu(z_0) < 0 \text{ and } | P_\nu(z_0) | > | P_{\nu-1}(z_0) |$$

Hence by the method of proof used in § 5, I. we have

$$| P_{\nu-1}(z_0) | < | P_\nu(z_0) | < | P_{\nu+1}(z_0) | < \cdots\cdots.$$

Therefore

$$P_{\nu+1}(z_0) = \lambda_{\nu+1}(z_0)\, P_\nu(z_0) - P_{\nu-1}(z_0) > 0,$$
$$P_{\nu+2}(z_0) = \lambda_{\nu+2}(z_0)\, P_{\nu+1}(z_0) - P_\nu(z_0) < 0,$$
$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots.$$

Similarly, if $P_{\nu-2}(z_0) < 0$ and $P_{\nu-1}(z_0) < 0$, then

$$P_\nu(z_0) > 0,\quad P_{\nu+1}(z_0) < 0,\quad P_{\nu+2}(z) > 0,\ \cdots\cdots.$$

Next since the roots of the equations

$$P_1(z)=0,\ P_2(z)=0,\ \cdots\cdots,\ P_\nu(z)=0,\ \cdots\cdots$$

form an *enumerable set* M, and since the set M is not perfect, we can take a real number z, which does not belong to M, in any finite interval.

Accordingly, *if z_0 be a positive number which does not belong to the set M, we can choose a positive integer N, corresponding to z_0, such that the polynomials*

$$P_N(z_0),\ P_{N+1}(z_0),\ P_{N+2}(z_0), \cdots\cdots$$

have different signs alternately.

Now let

$$V_N(z_0) = \mu, \qquad N \geqq \mu \geqq 0.$$

Then

$$V_{N+1}(z_0) = \mu+1, \quad V_{N+2}(z_0) = \mu+2,\ \cdots\cdots.$$

But since

$$V_N(-\infty) = N,\ V_{N+1}(-\infty) = N+1,\ \cdots\cdots,$$

we have

$$V_{N+k}(-\infty) - V_{N+k}(z_0) = N - \mu,\ (k=0, 1, 2, \cdots\cdots).$$

Therefore *the number of the roots of $P_\nu(z)=0$ $(\nu \geqq N)$ lying in the interval $-\infty < z < z_0$ is always same, however great ν $(\nu \geqq N)$ may be.*

Number the roots of $P_\nu(z)=0$ $(\nu \geqq N)$ in natural order beginning with the smallest and denote by $r_{\nu,i}$ the ith root $(1 \leqq i \leqq \nu)$ such that

$$r_{\nu,\,i} > z_0 > 0.$$

Then, as we have just proved,

$$r_{\nu+1,\,i} > z_0, \quad r_{\nu+2,\,i} > z_0, \cdots\cdots.$$

Moreover since, by § 6, the $\nu+1$ roots of $P_{\nu+1}(z)=0$ are separated by the ν roots of $P_\nu(z)=0$, we must have

$$r_{\nu,\,i} > r_{\nu+1,\,i} > r_{\nu+2,\,i} > \cdots\cdots > z_0 > 0.$$

Hence *the i th roots of*

$$P_\nu(z)=0, \quad P_{\nu+1}(z)=0, \cdots\cdots, \quad P_{\nu+k}(z)=0, \cdots\cdots$$

decrease monotonously as k increases, and approach a limiting value (r_i, say) which is positive, as k becomes infinite.

8. Since

$$P_\nu(r_{\nu,\,i})=0, \quad Q_\nu(r_{\nu,\,i})\neq 0, \quad (\nu=N, N+1, \cdots\cdots),$$

we have

$$K_\nu(r_{\nu,\,i})=\frac{P_\nu(r_{\nu,\,i})}{Q_\nu(r_{\nu,\,i})}=0.$$

We have already shown that the sequence $K_\nu(z)$, $(\nu=N, N+1, \cdots\cdots)$, converges uniformly to $K(z)$ when z does not lie within the vicinity of ζ_j, $(j=1, 2, \cdots\cdots)$. On the other hand, since ζ_j is a pole of $K(z)$, there exists a positive quantity ε such that

$$|\,K(z)\,| > G, \quad |\,z-\zeta_j\,| < \varepsilon,$$

however great G may be. Hence in this case we can choose the positive integer N' such that

$$|\,K_\nu(z)\,| > G-\delta, \quad \delta>0, \quad \nu \geqq N' > N,$$

however small δ may be.

It follows from

$$K_\nu(r_{\nu,\,i})=0, \quad \nu \geqq N'$$

that the roots $r_{\nu,\,i}$ $(\nu=N', N'+1, \cdots\cdots)$ do not lie in the vicinity of the pole ζ_j; whence *the sequence*

$$K_\nu(z), \quad (\nu=N', N'+1, \cdots\cdots)$$

converges uniformly to $K(z)$ in the vicinity of

$$z=r_{\nu,\,i}, \quad (\nu=N', N'+1, \cdots\cdots)$$

Therefore, by the well known theorem due to Hurwitz[1], the roots of $K(z)$ are identical with the limiting values of the roots of $K_\nu(z)$, $(\nu=N', N'+1, \cdots\cdots)$. Consequently r_i is a positive root of $K(z)$; and similarly for $r_{i+1}, r_{i+2}, \cdots\cdots, r_\nu$. But since ν may be taken as great as we please, we arrive at the theorem:

Theorem II. *The continued fraction* (1) *has infinitely many positive roots.*

Ikeda near Ôsaka, September 1919.

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(1) A. Hurwitz, "Über die Nullstellen der Bessel'schen Funktion," Math. Ann., 33 (1889), p. 246.

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